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GROUP VELOCITY AND POWER FLOW RELATIONS FOR SURFACE WAVES IN PLANE-STRATIFIED ANISOTROPIC MEDIA

by

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MRI

POLYTECHNIC INSTITUTE OF BROOKLYN
MICROWAVE RESEARCH INSTITUTE
Electrophysics Department

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H. L. Bertoni and A. Hessel

Polytechnic Institute of Brooklyn
Electrophysics Department
Graduate Center
Route 110
Farmingdale, New York 11735

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H. L. Bertoni

A. Hessel

Herbert J. Carlin
Head of Department

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Office of Naval Research
Washington, D. C.

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SUMMARY

Two aspects of power flow associated with electromagnetic waves in plane-stratified, dispersive, anisotropic media that are also lossless and linear are considered. One aspect is the relation between group velocity and the velocity of energy transport of surface waves in such media. It is shown that the group velocity of surface waves is equal to the ratio of the real part of the complex Poynting vector, integrated over the coordinate of stratification, to the corresponding integral of the stored energy density. The second aspect is the relation between the dyadic surface impedance representing either a slab of plane-stratified medium above a perfectly conducting plane or a semi-infinite region, the latter for the case of evanescent fields, and the power flow in the respective structures. The significance of the surface impedance and power relations for surface waves is discussed.

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INTRODUCTION

In this paper two aspects of the power flow associated with electromagnetic waves in plane-stratified, anisotropic, dispersive media and their application to surface wave propagation are considered. The first aspect is that of the relation between the group velocity and the velocity of energy transport of surface waves; the second is the relation between a dyadic surface impedance and the power flow and stored energy in the structure it represents.

It is well known that monochromatic plane electromagnetic waves in a homogeneous, dispersive, anisotropic medium that is also lossless and linear, e. g., the ionosphere for small-signal propagation, carry power in the direction of the normal to the plane wave dispersion surface. Specifically, the velocity of energy transport of plane waves in such a medium is equal to the group velocity, that is, the gradient in the wave number space of the frequency.[1, 2] This relation between the group velocity and the velocity of energy transport finds an important application in the ray interpretation of the far fields radiated by sources in the presence of homogeneous anisotropic media.[3]

It is shown here that an analogous relation involving the group velocity holds for the case of surface waves in plane-stratified, dispersive, anisotropic media that are also lossless and linear, in that the group velocity of surface waves that can propagate in such a medium may be interpreted as the surface wave energy transport velocity. For such surface waves, the direction as well as the magnitude of the real part $\underline{s}$ of the complex Poynting vector is, in general, a function of z, the coordinate in the direction of stratification. (An example of this dependence is described in Reference (4)). Therefore, $\underline{s}$ divided by the energy density cannot be identically equal to the surface wave group velocity, which is a vector independent of z. It will be shown, however, that the group velocity of the surface wave is identical to the velocity of energy transport of the surface wave taken as a whole, i. e., the gradient in the transverse wave number plane of the frequency is equal to the integral over z of $\underline{s}$ divided by the corresponding integral over z of the stored energy density. In a manner analogous to that for plane waves in anisotropic, homogeneous media, the relation between group velocity and energy transport velocity for surface waves should prove useful in formulating a ray interpretation for the surface wave fields excited by a source.[5, 6]

The proof of the relation between the group velocity and the energy transport velocity is furnished for two configurations. In the first section of this paper, the case

considered is that of a plane-stratified medium filling all space, while the second section contains the proof for the case of a plane-stratified medium filling the half-space above a perfectly conducting plane at $z = 0$.

In the third section of this paper, the relation between the dyadic surface impedance at $z = d$, which represents a plane-stratified, lossless, anisotropic medium filling the region $0 < z < d$ and bounded at $z = 0$ by a perfectly conducting plane, and the power flow and stored energy in this region is considered. With the help of the developments of the first section, it is shown that the power flow and stored energy in the region $0 < z < d$ are directly related to the derivatives of the surface impedance, with respect to transverse wave numbers and frequency, and to the components of the r. f. magnetic field transverse to z at z=d. The relation of power flow and energy to the surface reactance apply for all frequencies and real transverse wave numbers, not just those associated with surface waves. In particular, for surface waves propagating above a dyadic impedance plane, the power flow and energy relations are shown to be significant in calculating the energy transport velocity.

The fourth section of this paper is devoted to a discussion of the dyadic surface impedance representation of a semi-infinite, plane-stratified, lossless, anisotropic medium for ranges of frequency and transverse wave numbers for which the fields in the medium are evanescent at infinity. Again the power flow and stored energy relation involving the surface impedance are obtained. The power division between the space inside and outside of a surface wave guiding structure is determined in terms of the dyadic surface impedances defined in the third and fourth sections of the paper.

The Appendix treats briefly the dyadic admittance representation of a medium above a perfectly conducting plane. At those values of frequency and transverse wave numbers where the impedance formalism breaks down, the admittance formalism may, in general, still be used. Power flow and stored energy relations in terms of the surface admittance are given.

STRATIFIED MEDIUM FILLING ALL SPACE

A lossless, anisotropic, dispersive, plane-stratified medium is assumed to fill all space. It is uniform in the x and y directions and its interaction with a monochromatic electromagnetic field can be described in terms of the constitutive parameters of the medium, the dielectric tensor $\tilde{\epsilon}$ and the permeability tensor $\tilde{\mu}$. Since the medium is lossless, $\tilde{\epsilon}$ and $\tilde{\mu}$ are Hermitian,[2, 7, 8] and because of the assumed uniformity in x

and y, they are independent of these coordinates. The tensors $\underset{\sim}{\epsilon}$ and $\underset{\sim}{\mu}$ are analytic functions of the angular frequency ω and are assumed to be continuous functions of z except for a possibly denumerable number of finite jumps. The z dependence of $\underset{\sim}{\epsilon}$ and $\underset{\sim}{\mu}$ is further assumed to be such that the medium supports surface waves propagating transversely to z. Such surface waves are solutions of the source-free Maxwell equations and have the form

$$\left.\begin{matrix} \underline{E}(\underline{r};\ \underline{k}_t,\ \omega) \\ \\ \underline{H}(\underline{r};\ \underline{k}_t,\ \omega) \end{matrix}\right\} = \left\{\begin{matrix} \underline{e}(z;\ \underline{k}_t,\ \omega) \\ \\ \underline{h}(z;\ \underline{k}_t,\ \omega) \end{matrix}\right\} e^{-j\underline{k}_t \cdot \underline{\rho}} \tag{1}$$

where $\underline{e}$ and $\underline{h}$ tend to zero as $|z|$ approaches infinity. As used throughout this paper $\underline{\rho} = \underline{x}_o x + \underline{y}_o y$ is the position vector transverse to z and $\underline{k}_t = \underline{x}_o k_x + \underline{y}_o k_y$ is a real transverse wave vector. The vector amplitudes $\underline{e}$ and $\underline{h}$ are required to be such that the electric and magnetic energy densities, as well as $\mathrm{Re}(\underline{e} \times \underline{h}^*)$, are integrable on the infinite interval $-\infty < z < \infty$. Because the dependence of $\underline{E}$ and $\underline{H}$ on x and y is $e^{-j\underline{k}_t \cdot \underline{\rho}}$, the electric and magnetic energy densities and $\underline{E} \times \underline{H}^*$ are independent of $\underline{\rho}$. Furthermore, since the field components transverse to z, $\underline{e}_t$ and $\underline{h}_t$, must be continuous in z across any jump in $\underset{\sim}{\epsilon}$ or $\underset{\sim}{\mu}$, they must be continuous functions of z. For simplicity, $\underline{e}$ and $\underline{h}$ are assumed to be Rms quantities.

In the absence of sources, Maxwell's equations in a medium described by $\underset{\sim}{\epsilon}$ and $\underset{\sim}{\mu}$ are

$$\left.\begin{aligned} \nabla \times \underline{H} &= j\omega\underset{\sim}{\epsilon} \cdot \underline{E} \\ \nabla \times \underline{E} &= -j\omega\underset{\sim}{\mu} \cdot \underline{H} \end{aligned}\right\} \tag{2}$$

where the harmonic time dependence $e^{j\omega t}$ has been suppressed. Substituting $\underline{E}$ and $\underline{H}$ from equation (1) into (2) results in six linear homogeneous equations in the six unknown field components, four ordinary differential equations in the variable z and two algebraic equations. For any particular medium that can support surface waves, these six equations will have solutions satisfying the cavity-type boundary conditions $\underline{e} = \underline{h} = 0$ at $|z| = \infty$, and possessing the integrability properties described above, only for restricted values of the parameters $\underline{k}_t$ and ω that satisfy some functional relation of the form

$$D_s(\underline{k}_t, \omega) = 0 \tag{3}$$

where in general D_s is a regular function of $\underline{k}_t$ and ω. Relation (3) is the surface wave dispersion relation and determines the possible surface waves that can propagate transversely to z in the particular plane-stratified medium.

Let $\underline{k}_t$ and ω be such as to satisfy the surface wave dispersion relation (3) and consider neighboring values $\underline{k}_t + d\underline{k}_t$ and $\omega + d\omega$, also satisfying (3). The fields of that surface wave propagating with wave vector $\underline{k}_t + d\underline{k}_t$ at the frequency $\omega + d\omega$ are given, to first order in differential quantities, by

$$\left.\begin{aligned} \underline{E}(\underline{r}; \underline{k}_t + d\underline{k}_t, \omega + d\omega) &= \underline{E}(\underline{r}; \underline{k}_t, \omega) + \delta\underline{E}(\underline{r}; \underline{k}_t, \omega) \\ \underline{H}(\underline{r}; \underline{k}_t + d\underline{k}_t, \omega + d\omega) &= \underline{H}(\underline{r}; \underline{k}_t, \omega) + \delta\underline{H}(\underline{r}; \underline{k}_t, \omega) \end{aligned}\right\} \tag{4}$$

where the variation δ symbolizes the differential operation

$$\delta = d\underline{k}_t \cdot \nabla_{\underline{k}_t} + d\omega \frac{\partial}{\partial\omega} , \tag{5}$$

with $\nabla_{\underline{k}_t} = \underline{x}_o \frac{\partial}{\partial k_x} + \underline{y}_o \frac{\partial}{\partial k_y}$, and the partial derivatives of $\underline{E}$ and $\underline{H}$ are evaluated at $(\underline{k}_t, \omega)$. Since $\underline{E}_t$ and $\underline{H}_t$ are continuous functions of z for both sets of values $(\underline{k}_t, \omega)$ and $(\underline{k}_t + d\underline{k}_t, \omega + d\omega)$, it is seen from (4) that the variations $\delta\underline{E}_t$ and $\delta\underline{H}_t$ must also be continuous functions of z. The differential equations that $\delta\underline{E}$ and $\delta\underline{H}$ satisfy can be found by applying the variation δ to Maxwell's equations. Recalling (5) and because $\underset{\sim}{\epsilon}$ and $\underset{\sim}{\mu}$ do not depend on $\underline{k}_t$, the variation on Maxwell's equations results in

$$\left.\begin{aligned} \nabla \times \delta\underline{H} &= j\, d\omega \frac{\partial \omega \underset{\sim}{\epsilon}}{\partial\omega} \cdot \underline{E} + j\omega \underset{\sim}{\epsilon} \cdot \delta\underline{E} \\ \nabla \times \delta\underline{E} &= j\, d\omega \frac{\partial \omega \underset{\sim}{\mu}}{\partial\omega} \cdot \underline{H} - j\omega \underset{\sim}{\mu} \cdot \delta\underline{H} \end{aligned}\right\} . \tag{6}$$

Consider now the identity

$$\begin{aligned} &\nabla \cdot (\underline{E}^* \times \delta\underline{H} + \delta\underline{E} \times \underline{H}^*) \\ &= \delta\underline{H} \cdot \nabla \times \underline{E}^* - \underline{E}^* \cdot \nabla \times \delta\underline{H} + \underline{H}^* \cdot \nabla \times \delta\underline{E} - \delta\underline{E} \cdot \nabla \times \underline{H}^* . \end{aligned} \tag{7}$$

With the help of equations (2) and (6), the right-hand side of (7) can be rewritten to give the relation

$$\nabla \cdot (\underline{E}^* \times \delta\underline{H} + \delta\underline{E} \times \underline{H}^*) = -j\,d\omega\left(\underline{E}^* \cdot \frac{\partial\omega\tilde{\varepsilon}}{\partial\omega} \cdot \underline{E} + \underline{H}^* \cdot \frac{\partial\omega\tilde{\mu}}{\partial\omega} \cdot \underline{H}\right) \tag{8}$$

when the assumption that $\tilde{\varepsilon}$ and $\tilde{\mu}$ are Hermitian is used to write $\delta\underline{E} \cdot \tilde{\varepsilon}^* \cdot \underline{E}^*$ as $\underline{E}^* \cdot \tilde{\varepsilon} \cdot \delta\underline{E}$ and $\delta\underline{H} \cdot \tilde{\mu}^* \cdot \underline{H}^*$ as $\underline{H}^* \cdot \tilde{\mu} \cdot \delta\underline{H}$. The first term on the right-hand side of (8) is twice the time average electric energy density w_e while the second is twice the time average magnetic energy density w_h.[2,8,9] As pointed out, w_e and w_h are independent of $\underline{\rho}$. In terms of the total energy density $w = w_e + w_h$, equation (8) can then be written

$$\nabla \cdot (\underline{E}^* \times \delta\underline{H} + \delta\underline{E} \times \underline{H}^*) = -j\,2\,w\,d\omega \ . \tag{9}$$

The divergence term on the left-hand side of (9) is now evaluated by expanding $\delta\underline{E}$ and $\delta\underline{H}$ and subsequently applying the ∇ operator. With the help of (5) and (1), it is seen that

$$\left.\begin{aligned} \delta\underline{E} &= (\delta\underline{e} - j\,d\underline{k}_t \cdot \underline{\rho}\ \underline{e})\, e^{-j\underline{k}_t \cdot \underline{\rho}} \\ \delta\underline{H} &= (\delta\underline{h} - j\,d\underline{k}_t \cdot \underline{\rho}\ \underline{h})\, e^{-j\underline{k}_t \cdot \underline{\rho}} \end{aligned}\right\} . \tag{10}$$

The variations on the transverse vector amplitudes, $\delta\underline{e}_t$ and $\delta\underline{h}_t$, are continuous functions of z since $\delta\underline{E}_t$, $\delta\underline{H}_t$, $\underline{e}_t$ and $\underline{h}_t$ are, a fact that will prove useful later. Since $\underline{e}$ and $\underline{h}$ are independent of $\underline{\rho}$, using the above relations one has

$$\begin{aligned} &\nabla \cdot (\underline{E}^* \times \delta\underline{H} + \delta\underline{E} \times \underline{H}^*) \\ &= \nabla \cdot [\underline{e}^* \times \delta\underline{h} + \delta\underline{e} \times \underline{h}^* - j\,d\underline{k}_t \cdot \underline{\rho}\,(\underline{e}^* \times \underline{h} + \underline{e} \times \underline{h}^*)] \\ &= \frac{\partial}{\partial z}\,\underline{z}_o \cdot (\underline{e}^* \times \delta\underline{h} + \delta\underline{e} \times \underline{h}^*) - j\,2\,d\underline{k}_t \cdot \underline{\rho}\,\frac{\partial}{\partial z}(\underline{z}_o \cdot \underline{s}) - j\,2\,d\underline{k}_t \cdot \underline{s} \end{aligned} \tag{11}$$

where $\underline{s}(z) = \mathrm{Re}(\underline{e} \times \underline{h}^*)$ is the real part of the complex Poynting vector $\underline{E} \times \underline{H}^* = \underline{e} \times \underline{h}^*$. It is easily seen that the term $\underline{z}_o \cdot \underline{s}$ is independent of z, i.e., $\frac{\partial}{\partial z}(\underline{z}_o \cdot \underline{s}) = 0$, since in a source-free region filled with a lossless medium, the divergence of the real part of the Poynting vector is zero and, for the plane-stratified medium under discussion, $\underline{s}$ is

independent of ω so that $\frac{\partial}{\partial x}(\underline{x}_o \cdot \underline{s}) = \frac{\partial}{\partial y}(\underline{y}_o \cdot \underline{s}) = 0.$[†] Thus with the aid of (11), equation (9) becomes

$$j\frac{1}{2}\,\frac{\partial}{\partial z}\,\underline{z}_o \cdot (\underline{e}^* \times \delta\underline{h} + \delta\underline{e} \times \underline{h}^*) + d\underline{k}_t \cdot \underline{s} = w\,d\omega \quad . \tag{12}$$

In the derivation of (12), the essential assumptions used are that the medium be lossless and that it be plane-stratified so that waves of the form given in (1) satisfy the source-free Maxwell equations. The assumption that $(\underline{k}_t, \omega)$ and $(\underline{k}_t + d\underline{k}_t, \omega + d\omega)$ satisfy the surface wave dispersion relation (3) serves to restrict the changes $d\underline{k}_t$ and $d\omega$ in $\underline{k}_t$ and ω to a surface in $\underline{k}_t - \omega$ space, so that $\delta\underline{e}_t$ and $\delta\underline{h}_t$ will be continuous functions of z for all $-\infty < z < \infty$ and tend to zero as $|z| \to \infty$.

Since the wave vector $\underline{k}_t + d\underline{k}_t$ and the frequency $\omega + d\omega$ satisfy the surface wave dispersion relation, to first order $d\omega = d\underline{k}_t \cdot \nabla_{\underline{k}_t}\omega(\underline{k}_t)$ where $\omega(\underline{k}_t)$ is the solution of the dispersion relation (3). Using this expression for $d\omega$, and after rearranging, equation (12) becomes

$$j\frac{1}{2}\,\frac{\partial}{\partial z}\,\underline{z}_o \cdot (\underline{e}^* \times \delta\underline{h} + \delta\underline{e} \times \underline{h}^*) = d\underline{k}_t \cdot (w\nabla_{\underline{k}_t}\omega - \underline{s}) \quad . \tag{13}$$

The term on the left of (13) does not vanish identically so that in general $\underline{s} \neq w\nabla_{\underline{k}_t}\omega$ and hence in the surface wave case $\nabla_{\underline{k}_t}\omega$ cannot be interpreted as a local energy velocity.

In order to eliminate the term on the left-hand side of (13) we now integrate this relation and obtain

$$j\frac{1}{2}\int_{-\infty}^{\infty} \frac{\partial}{\partial z}\,\underline{z}_o \cdot (\underline{e}^* \times \delta\underline{h} + \delta\underline{e} \times \underline{h}^*)\,dz = d\underline{k}_t \cdot (W\nabla_{\underline{k}_t}\omega - \underline{S}) \tag{14}$$

where

$$\underline{S} = \int_{-\infty}^{\infty} \underline{s}\,dz \tag{15}$$

and

$$W = \int_{-\infty}^{\infty} w\,dz \quad . \tag{16}$$

Because $s_z = \underline{z}_o \cdot \underline{s}$ is independent of z, as discussed above, and is zero at $|z| = \infty$, since $\underline{e}$ and $\underline{h}$ are zero there, s_z is zero for all values of z. Thus $\underline{s}$, and hence $\underline{S}$,

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[†]Because $\underline{e}_t$ and $\underline{h}_t$ are continuous functions of z, $\underline{z}_o \cdot \underline{s}$ is a continuous function of z and therefore $\frac{\partial}{\partial z}(\underline{z}_o \cdot \underline{s})$ cannot have a delta function behavior at the jumps of $\underset{\sim}{\epsilon}$ or $\underset{\sim}{\mu}$.

are purely transverse vectors.

Recognizing that

$$\underline{z}_o \cdot (\underline{e}^* \times \delta\underline{h} + \delta\underline{e} \times \underline{h}^*) = \underline{z}_o \cdot (\underline{e}_t^* \times \delta\underline{h}_t + \delta\underline{e}_t \times \underline{h}_t^*) \quad , \tag{17}$$

and using the fact that $\underline{e}_t$, $\underline{h}_t$, $\delta\underline{e}_t$ and $\delta\underline{h}_t$ are continuous functions of z, one has

$$\int_{-\infty}^{\infty} \frac{\partial}{\partial z} \underline{z}_o \cdot (\underline{e}^* \times \delta\underline{h} + \delta\underline{e} \times \underline{h}^*)\, dz = \underline{z}_o \cdot (\underline{e}_t^* \times \delta\underline{h}_t + \delta\underline{e}_t \times \underline{h}_t^*)\Big|_{-\infty}^{\infty} \quad , \tag{18}$$

which vanishes as a consequence of the boundary conditions on $\underline{e}$ and $\underline{h}$ at $|z| = \infty$. Hence,

$$d\underline{k}_t \cdot (W\nabla_{\underline{k}_t}\omega - \underline{S}) = 0 \tag{19}$$

and, since $\underline{S}$ is a purely transverse vector and $d\underline{k}_t$ is arbitrary, it follows that

$$\nabla_{\underline{k}_t}\omega = \underline{S}/W \quad . \tag{20}$$

Although the real Poynting vector $\underline{s}$ can vary in magnitude and direction with z, the total real Poynting vector $\underline{S}$ is independent of z and represents the total surface wave power flow across a strip normal to $\underline{S}$, infinite in z and having unit width. The term W represents the total stored energy of the surface wave fields in an infinite cylinder, parallel to z, whose x-y cross section has unit area. Equation (20) thus states that the group velocity of the surface wave, $\nabla_{\underline{k}_t}\omega$, is equal to the velocity of energy transport $\underline{S}/W$ of the surface wave as a whole. This statement for the surface waves in plane-stratified media replaces the relation $\nabla_{\underline{k}}\omega = \underline{s}/w$ for plane waves in homogeneous anisotropic media and should be useful in the ray interpretation of the surface wave fields excited by a source in anisotropic plane-stratified media.

STRATIFIED MEDIUM ABOVE A PERFECTLY CONDUCTING PLANE

The plane-stratified medium described in the first section is now assumed to fill the half-space above a perfectly conducting plane at z = 0. Again we assume the z dependence of $\underset{\sim}{\epsilon}$ and $\underset{\sim}{\mu}$ to be such that surface waves of the form given in (1) can propagate transversely to the direction of stratification. The vector amplitudes $\underline{e}$ and $\underline{h}$ of these waves tend to zero as z approaches infinity and satisfy the boundary condition

$\underline{e}_t = 0$ at $z = 0$. The electric and magnetic energy densities, as well as $\mathrm{Re}(\underline{e} \times \underline{h}^*)$, are now assumed to be integrable on the semi-infinite interval $0 < z < \infty$. As discussed in the previous section, $\underline{e}_t$ and $\underline{h}_t$ must be continuous functions of z. Solutions of Maxwell's equations satisfying the above conditions occur only for values of $\underline{k}_t$ and ω that obey a surface wave dispersion relation, $D_s(\underline{k}_t, \omega) = 0$, valid for the semi-infinite medium.

As in the previous section, the fields at two neighboring sets of values, $(\underline{k}_t, \omega)$ and $(\underline{k}_t + d\underline{k}_t, \omega + d\omega)$, both of which satisfy the dispersion relation, are considered. Using equation (4), the fields $\underline{E}$ and $\underline{H}$ at $(\underline{k}_t + d\underline{k}_t, \omega + d\omega)$ are found, to first order, in terms of the fields and their derivatives, with respect to k_x, k_y and ω, evaluated at $(\underline{k}_t, \omega)$. Since the variations in the fields, $\delta\underline{E}$ and $\delta\underline{H}$, in this problem also satisfy (6), equation (13) holds in this case as well. Because the term on the left-hand side of (13) is, in general, not zero, $\nabla_{\underline{k}_t}\omega$ again cannot be interpreted as a local surface wave energy velocity. However, upon integration of (13) over the interval $0 < z < \infty$, the left-hand side vanishes and $\nabla_{\underline{k}_t}\omega$ can again be interpreted as the velocity of energy transport of the surface wave as a whole. To see this, one recognizes that since $\underline{e}_t = 0$ at $z = 0$, $\delta\underline{e}_t$ must also be zero there. Hence, using equation (17) and recalling that $\underline{e}$ and $\underline{h}$ are zero at $z = \infty$, it is seen that

$$\int_0^\infty \frac{\partial}{\partial z}\, \underline{z}_o \cdot (\underline{e}^* \times \delta\underline{h} + \delta\underline{e} \times \underline{h}^*)\, dz = \underline{z}_o \cdot (\underline{e}_t^* \times \delta\underline{h}_t + \delta\underline{e}_t \times \underline{h}_t^*)\Big|_0^\infty = 0 \quad . \qquad (21)$$

Defining

$$W = \int_0^\infty w\, dz \qquad (22)$$

and

$$\underline{S} = \int_0^\infty \underline{s}\, dz \qquad (23)$$

($\underline{S}$ being a purely transverse vector since $s_z = 0$) the integration of (13) over the interval $0 < z < \infty$ gives, in view of (21),

$$0 = d\underline{k}_t \cdot (W\nabla_{\underline{k}_t}\omega - \underline{S}) \quad . \qquad (24)$$

Again, because $d\underline{k}_t$ is arbitrary, it follows that

$$\nabla_{\underline{k}_t}\omega = \underline{S}/W \quad . \qquad (25)$$

That is to say, for surface waves above a perfectly conducting plane, the group velocity v_{k_t} is equal to the velocity of energy transport $\underline{S}/W$ of the surface wave as a whole.[†]

SURFACE IMPEDANCE AND POWER FLOW RELATIONS

When formulating steady-state electromagnetic problems involving fields of the form given in (1) in a lossless, plane-stratified anisotropic medium above a perfectly conducting plane at $z = 0$, it is sometimes profitable to represent the effect of the structure below a plane $z = d > 0$ on the fields in the region $z > d$ by a surface impedance dyadic at $z = d$. The impedance dyadic $\underset{\sim}{Z}$ may then be employed as an equivalent boundary condition at $z = d$ when solving for the fields in the region $z > d$. In this section the relation between the derivatives of the impedance dyadic, with respect to the spatial wave numbers k_x and k_y, and the power flowing in the region $0 < z < d$ will be established and the significance of this relation for surface waves supported by such an equivalent impedance plane will be pointed out. The relation between $\partial \underset{\sim}{Z}/\partial\omega$ and stored energy in the region $0 < z < d$ will also be established.

In order to define $\underset{\sim}{Z}$ and to find its relation to power flow and stored energy in the region $0 < z < d$, consideration is first given to the auxiliary problem of finding the fields in this region when $\underline{H}_t$ of the form given in (1) is specified at $z = d$. Thus, we look in the region $0 < z < d$ for the solution of Maxwell's equations that satisfies the boundary conditions

$$\underline{E}_t = 0 \tag{26}$$

at $z = 0$ and

$$\underline{H}_t(\underline{r}, t) = \underline{h}_d e^{j(\omega t - \underline{k}_t \cdot \underline{\rho})} \tag{27}$$

at $z = d$. All values of $\underline{k}_t$ and ω, except those at which $\underset{\sim}{Z}$ is singular, are considered (for further discussion, see the Appendix). No restrictions are placed on the fields in the region $z > d$. In fact, the medium filling the region above the plane $z = d$ may be taken to be arbitrarily stratified, since, with $\underline{H}_t$ rigidly prescribed at $z = d$, the medium does

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[†] If a second perfectly conducting plane at $z = d > 0$ is present, it is easily seen that (25) is still valid for the fields between the conducting planes if $\underline{S}$ and W are now taken as $\underline{S} = \int_0^d \underline{s}\, dz$ and $W = \int_0^d w\, dz$. Thus for waves in a parallel plate wave guide filled with a plane-stratified, lossless, anisotropic medium, the group velocity is equal to the velocity of energy transport.

not affect the fields for $0 < z < d$. The medium filling the region $0 < z < d$ is assumed to be lossless, uniform in x and y and characterized by $\underset{\sim}{\epsilon}$ and $\underset{\sim}{\mu}$, which are analytic functions of ω.

Specification of the above boundary conditions is sufficient, in general, to uniquely determine the fields, and hence the power flow and stored energy, in the region $0 < z < d$. Solving this auxiliary problem for arbitrary polarizations of $\underline{h}_d$ then permits a unique determination of the dyadic surface impedance $\underset{\sim}{Z}$. Having determined $\underset{\sim}{Z}$ from the auxiliary problem, one can now solve for the fields above the impedance plane $z = d$ in terms of $\underset{\sim}{Z}$, the excitation in the region $z > d$, and the boundary conditions at $z = \infty$. The requirement that $\underline{H}_t$ be continuous across $z = d$ now permits one to uniquely determine the fields, and thus the power flow and stored energy, in the region $0 < z < d$ in terms of $(\underline{H}_t)_{z=d^+}$ and the given $\underset{\sim}{Z}$.

In practice, the auxiliary problem need be solved for only two linearly independent polarizations of $\underline{h}_d$, since the linearity of Maxwell's equations permits the solution for any other polarization of $\underline{h}_d$ to be expressed in terms of those for the two independent polarizations. Thus, at each of the values of $\underline{k}_t$ and ω to be considered, we solve for the fields in the region $0 < z < d$ when $\underline{h}_d$ takes on two linearly independent polarizations, e.g., $\underline{h}_d = \underline{x}_o$ and $\underline{h}_d = \underline{y}_o$. Having found the fields, which will be of the form given in (1), for both polarizations of $\underline{h}_d$, $\underset{\sim}{Z}$ may uniquely be defined by requiring that the relation

$$(\underline{E}_t)_{z=d} = \underset{\sim}{Z} \cdot (\underline{z}_o \times \underline{H}_t)_{z=d} \tag{28}$$

be satisfied for both sets of fields. This requirement is equivalent to specifying four inhomogeneous, linearly independent equations from which the four unknown elements of $\underset{\sim}{Z}$ can be found. If one now wishes to solve for fields of the form given in (1), in the region $z > d$, relation (28) may be used as a boundary condition at $z = d$, which will ensure that the transverse fields connect continuously to valid fields in the region $0 < z < d$. That is, if $\underline{e}$ and $\underline{h}$ in the region $z > d$ are such that (28) is satisfied, then, taking $\underline{h}_d$ as $(\underline{h}_t)_{z=d^+}$, the corresponding $\underline{e}$ in the region $0 < z < d$ will be such that $(\underline{e}_t)_{z=d^-} = (\underline{e}_t)_{z=d^+}$.

Having thus defined $\underset{\sim}{Z}$, we proceed to investigate the meaning of its derivatives with respect to k_x, k_y and ω. To this end, we assume the fields in the region $0 < z < d$ are known and $\underline{h}_d$ of (27) has been selected such that the derivatives of the fields with

respect to k_x, k_y and ω exist. Equation (12) can now be employed where $d\underline{k}_t$ and $d\omega$ in the variation δ are arbitrary and independent. Equation (12) is valid for the fields in the region $0 < z < d$ since the assumptions used in deriving it are also satisfied in the present case — see the text after (12). The restrictions placed on $d\underline{k}_t$ and $d\omega$ in the first section are not necessary in the present discussion since, as previously mentioned, solutions of Maxwell's equations for which $\underline{e}_t$ and $\underline{h}_t$ are continuous in z will exist in the region $0 < z < d$ for all $\underline{k}_t$ and ω (excepting the singular points of Z, as discussed in the Appendix), thus ensuring that $\delta\underline{e}_t$ and $\delta\underline{h}_t$ are continuous functions of z for all $d\underline{k}_t$ and $d\omega$. Since the fields are bounded as z approaches d and as z approaches 0, it is permissible to integrate (12) over the closed interval $0 \leq z \leq d$. Performing the integration and using (17) yields the relation

$$j\frac{1}{2}\,\underline{z}_o \cdot (\underline{e}_t^* \times \delta\underline{h}_t + \delta\underline{e}_t \times \underline{h}_t^*)_{z=d} = W_d\, d\omega - d\underline{k}_t \cdot \underline{S}_d\,. \tag{29}$$

Here

$$W_d = \int_0^d w\, dz \tag{30}$$

and

$$\underline{S}_d = \int_0^d \underline{s}\, dz\,. \tag{31}$$

Note that since $\frac{\partial}{\partial z} s_z = \nabla \cdot \underline{s} = 0$ and $(s_z)_{z=0} = 0$, $s_z = 0$ for all $0 < z < d$ and hence $\underline{S}_d$ is a transverse vector.

Since $\underline{E}$ and $\underline{H}$ have the form given in (1), the impedance relation (28) may be rewritten as

$$(\underline{e}_t)_{z=d} = \underset{\sim}{Z} \cdot (\underline{z}_o \times \underline{h}_t)_{z=d}\,. \tag{32}$$

Applying the variation δ to the above equation gives

$$(\delta\underline{e}_t)_{z=d} = \underset{\sim}{Z} \cdot (\underline{z}_o \times \delta\underline{h}_t)_{z=d} + \delta\underset{\sim}{Z} \cdot (\underline{z}_o \times \underline{h}_t)_{z=d}\,. \tag{33}$$

Using $(\underline{e}_t)_{z=d}$ from (32) and $(\delta\underline{e}_t)_{z=d}$ from (33), it can be verified that

$$\underline{z}_o \cdot (\underline{e}_t^* \times \delta\underline{h}_t + \delta\underline{e}_t \times \underline{h}_t^*)_{z=d} = -[(\underline{z}_o \times \underline{h}_t^*) \cdot \delta\underset{\sim}{Z} \cdot (\underline{z}_o \times \underline{h}_t)]_{z=d} \tag{34}$$

when the anti-Hermitian property† of $\underset{\sim}{Z}$ is used to write $[(\underline{z}_o \times \underline{h}_t^*)\cdot \underset{\sim}{Z} \cdot (\underline{z}_o \times \delta \underline{h}_t)]_{z=d} = - [(\underline{z}_o \times \delta \underline{h}_t)\cdot \underset{\sim}{Z}^* \cdot (\underline{z}_o \times \underline{h}_t^*)]_{z=d}$. With relation (34), equation (29) becomes

$$-j\frac{1}{2}[(\underline{z}_o \times \underline{h}_t^*)\cdot \delta \underset{\sim}{Z} \cdot (\underline{z}_o \times \underline{h}_t)]_{z=d} = W_d\, d\omega - d\underline{k}_t \cdot \underline{S}_d \quad . \tag{35}$$

Since dk_x, dk_y and $d\omega$ are all independent, one finds that

$$\left.\begin{aligned} j\frac{1}{2}[(\underline{z}_o \times \underline{h}_t^*)\cdot \frac{\partial \underset{\sim}{Z}}{\partial k_x} \cdot (\underline{z}_o \times \underline{h}_t)]_{z=d} &= \underline{S}_{dx} \\ j\frac{1}{2}[(\underline{z}_o \times \underline{h}_t^*)\cdot \frac{\partial \underset{\sim}{Z}}{\partial k_y} \cdot (\underline{z}_o \times \underline{h}_t)]_{z=d} &= \underline{S}_{dy} \end{aligned}\right\} \tag{36}$$

and that

$$-j\frac{1}{2}[(\underline{z}_o \times \underline{h}_t^*)\cdot \frac{\partial \underset{\sim}{Z}}{\partial \omega} \cdot (\underline{z}_o \times \underline{h}_t)]_{z=d} = W_d \quad . \tag{37}$$

It is thus seen that W_d and $\underline{S}_d$ can be found knowing only $\underset{\sim}{Z}$ and $(\underline{h}_t)_{z=d}$. As previously pointed out, if fields of the form given in (1) exist in the region $z > d$ and satisfy the impedance boundary condition at $z = d$, there will be unique fields in the region $0 < z < d$ that satisfy the continuity conditions $(\underline{h}_t)_{z=d^-} = (\underline{h}_t)_{z=d^+}$ and $(\underline{e}_t)_{z=d^-} = (\underline{e}_t)_{z=d^+}$. Because of the continuity of $\underline{h}_t$ at $z = d$, the power flow and stored energy associated with the fields in the region $0 < z < d$ can be calculated from relations (36) and (37) using $(\underline{h}_t)_{z=d} = \lim_{z \downarrow d} \underline{h}_t$, i. e., the limit of $\underline{h}_t$ as z approaches d from above.

Since the relations (36) and (37) hold for arbitrary $\underline{k}_t$ and ω, they are valid, in particular, for values of $\underline{k}_t$ and ω that correspond to a surface wave. Thus relations (36) and (37), with appropriate values of $\underline{k}_t$ and ω, furnish an alternative way of calculating that portion of the surface wave power flowing in the slab and that portion of the stored energy of the surface wave which is in the slab.

†The impedance dyadic $\underset{\sim}{Z}$ is anti-Hermitian, i. e., the matrix representation for $\underset{\sim}{Z}$ has the property that the transpose conjugate $\underset{\sim}{Z}^+$ is equal to $-\underset{\sim}{Z}$. This property follows from the facts that $\varepsilon_z = 0$ for all $z < d$ and that the fields are continuous as z approaches d from below so that $\mathrm{Re}(\underline{e}_t \times \underline{h}_t^*)_{z=d}$ must be zero.[10]

The relation between power flow and the derivatives of $\underset{\sim}{Z}$ with respect to k_x and k_y given in (36) does not appear to have been previously recognized. While the connection between stored energy and $\partial \underset{\sim}{Z} / \partial \omega$, to the best of our knowledge, has not been shown explicitly for the case of traveling waves, the connection between stored power and the impedance matrix of a lossless junction is well known.[11]

The consistency of relations (36) and (37) for surface waves with the results obtained in the second section will now be shown. Consider a surface wave propagating in a lossless plane-stratified medium above a surface impedance plane at $z = d$. The surface impedance $\underset{\sim}{Z}$ is assumed to be known and to represent the effect of a plane-stratified, lossless medium above a perfectly conducting plane at $z = 0$. The surface wave fields in the region $z > d$ are assumed to be of the form given in (1) with $\underline{k}_t$ and ω related through the appropriate surface wave dispersion relation. The surface wave fields satisfy (13), which, when integrated over the interval $d < z < \infty$, yields the relation

$$-j\frac{1}{2}\underline{z}_o \cdot (\underline{e}_t^* \times \delta\underline{h}_t + \delta\underline{e}_t \times \underline{h}_t^*)_{z=d} = d\underline{k}_t \cdot [\nabla_{\underline{k}_t}\omega \int_d^\infty w\,dz - \int_d^\infty \underline{s}\,dz\,] \quad . \tag{38}$$

Since $\underline{e}_t$ and $\underline{h}_t$ satisfy the impedance condition (32), equation (34) holds. Using (34) and the fact that for the surface wave $d\omega = d\underline{k}_t \cdot \nabla_{\underline{k}_t}\omega$, the above equation can be written as

$$\begin{aligned} d\underline{k}_t \cdot \nabla_{\underline{k}_t}\omega \left\{ \int_d^\infty w\,dz - j\frac{1}{2}[(\underline{z}_o \times \underline{h}_t^*) \cdot \frac{\partial \underset{\sim}{Z}}{\partial \omega} \cdot (\underline{z}_o \times \underline{h}_t)]_{z=d} \right\} \\ = d\underline{k}_t \cdot \left\{ \int_d^\infty \underline{s}\,dz + \underline{x}_o \frac{j}{2}[(\underline{z}_o \times \underline{h}_t^*) \cdot \frac{\partial \underset{\sim}{Z}}{\partial k_x} \cdot (\underline{z}_o \times \underline{h}_t)]_{z=d} \right. \\ \left. + \underline{y}_o \frac{j}{2}[(\underline{z}_o \times \underline{h}_t^*) \cdot \frac{\partial \underset{\sim}{Z}}{\partial k_y} \cdot (\underline{z}_o \times \underline{h}_t)]_{z=d} \right\} \quad . \end{aligned} \tag{39}$$

As discussed above, the terms containing $\partial \underset{\sim}{Z} / \partial k_x$ and $\partial \underset{\sim}{Z} / \partial k_y$ that appear in (39) are equal to the x and y components of the power flow $\underline{S}_d$ below the plane $z = d$. Furthermore, the term containing $\partial \underset{\sim}{Z} / \partial \omega$ is equal to the stored energy, per unit area in the x-y plane, below the plane $z = d$, namely W_d. Thus (39) may be written

$$d\underline{k}_t \cdot \nabla_{\underline{k}_t}\omega \left\{ \int_d^\infty w\,dz + W_d \right\} = d\underline{k}_t \cdot \left\{ \int_d^\infty \underline{s}\,dz + \underline{S}_d \right\} \tag{40}$$

or, since $d\underline{k}_t$ is arbitrary,

$$\nabla_{\underline{k}_t}\omega \left\{ \int_d^\infty w\, dz + W_d \right\} = \int_d^\infty \underline{s}\, dz + \underline{S}_d \quad . \tag{41}$$

Finally, from the definition of W_d and $\underline{S}_d$ given in (30) and (31), equation (41) is seen to reduce to

$$\nabla_{\underline{k}_t}\omega \int_0^\infty w\, dz = \int_0^\infty \underline{s}\, dz \quad , \tag{42}$$

which is precisely the relation found to hold in the previous section for surface waves above a perfectly conducting plane at $z = 0$. Hence if a surface impedance boundary condition representing a plane-stratified medium above a perfectly conducting plane is used when solving for surface waves, the resultant group velocity $\nabla_{\underline{k}_t}\omega$ is equal to the energy transport velocity of the entire surface wave, not to just that portion of the surface wave above the impedance plane.

SURFACE IMPEDANCE FOR THE CASE OF EVANESCENT WAVES

In the derivation of the power flow and energy relations for a surface impedance representing a plane-stratified medium above a perfectly conducting plane, the presence of the conducting plane served to ensure that $s_z = 0$ and that the stored energy, per unit area in the x-y plane, and power flow are finite in the region $0 < z < d$ for all possible polarizations of $(\underline{h}_t)_{z=d}$ and all real values of $\underline{k}_t$ and ω. Since the fields of evanescent waves in a semi-infinite plane-stratified medium also possess these two properties, one would expect power flow and energy relations similar to (36) and (37) to exist in this case for the surface impedance representing the semi-infinite medium.

Let a semi-infinite, plane-stratified, lossless, dispersive, anisotropic medium fill the region above the plane $z = d$. By analogy to the case of the medium above a perfectly conducting plane, consideration is first given to the auxiliary problem of finding those fields in the region $z > d$ which satisfy the boundary condition

$$(\underline{H}_t)_{z=d} = \underline{h}_d e^{j(\omega t - \underline{k}_t \cdot \underline{\rho})} \tag{43}$$

at $z = d$ and the boundary conditions

$$\lim_{z \to \infty} (\underset{\sim}{E}, \underset{\sim}{H}) = 0 \quad . \tag{44}$$

In addition, we require that $\int_d^\infty \underline{s}\, dz$ and $\int_d^\infty w\, dz$ exist. The term "evanescent," as used in the rest of this paper, will refer to fields satisfying (44) and the foregoing integral requirements. Evanescent fields will also have the property that $s_z = 0$. The auxiliary problem is to be solved for all polarizations of $\underline{h}_d$ so that the surface impedance may be defined.

In general, only for limited regions in $\underline{k}_t - \omega$ space will the auxiliary problem have unique, non-trivial, evanescent solutions that satisfy (43) for all $\underline{h}_d$ and thus permit definition of the surface impedance $\underset{\sim}{Z}_s$. Other values of $\underline{k}_t$ and ω will not be considered for one of two reasons. First, in media having an appropriate z dependence, non-unique, cavity-type, evanescent solutions satisfying (43) with $\underline{h}_d = 0$ may exist for points lying on surfaces in $\underline{k}_t - \omega$ space. In such cases, $\underset{\sim}{Z}_s$ will have singularities on these surfaces. (Discussion of such points and the derivation of a surface admittance formalism that is, in general, regular at such points, are analogous to those given in the Appendix for a medium of finite thickness above a perfectly conducting plane.) Second, in some regions of $\underline{k}_t - \omega$ space, fields satisfying (43) will not be of the evanescent type for most or all polarizations of $\underline{h}_d$.† Thus, unless alternate boundary conditions are specified at $z = \infty$, such as the radiation condition, the fields, and hence $\underset{\sim}{Z}_s$, cannot be uniquely defined. Even if boundary conditions are impose at $z=\infty$ and if $\underset{\sim}{Z}_s$ is defined in this case (it is no longer anti-Hermitian), the associated fields do not possess the integration properties necessary to derive simple power flow and stored energy relations.

Hence only those regions in $\underline{k}_t - \omega$ space in which the auxiliary problem has unique, non-trivial solutions satisfying (43) and (44) for all $\underline{h}_d \neq 0$ will be considered here. The regions where the auxiliary problem can be solved, the nature of which depends on the particular medium under discussion, are assumed to exist and to form open sets, i.e., not merely surfaces, so that k_x, k_y, and ω will be continuous, independent variables within these regions.

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† An example of a region where no evanescent waves exist is formed by the points in and on the cone $\omega^2 = \frac{1}{\epsilon_0 \mu_0}(k_x^2 + k_y^2)$ when the medium being studied is free space. Outside this cone unique, non-trivial solutions of the auxiliary problem exist for all $\underline{h}_d$.

Thus restricting $\underline{k}_t$ and ω to those regions where unique, non-trivial solutions of the auxiliary problem are assumed to exist for all polarizations of $\underline{h}_d$, the surface impedance dyadic $\underset{\sim}{Z}_s$ can be defined for the semi-infinite region. Since the linearity of Maxwell's equations permits the solutions for all $\underline{h}_d$ to be expressed as a superposition of the solutions for two linearly independent polarizations of $\underline{h}_d$, we again need consider only two such polarizations, e. g., $\underline{h}_d = \underline{x}_o$ and $\underline{h}_d = \underline{y}_o$. From the solutions of the auxiliary problem, which will be of the form given in (1), for these two polarizations, $\underset{\sim}{Z}_s$ can be found uniquely from the requirement that the relation

$$(\underline{e}_t)_{z=d} = \underset{\sim}{Z}_s \cdot (-\underline{z}_o \times \underline{h}_t)_{z=d} \tag{45}$$

be satisfied by the fields of both solutions.[†]

By analogy to the discussion given in the previous section, $\underset{\sim}{Z}_s$ may be used as a boundary condition at $z = d$ when solving for the fields below this plane. Also, requiring $\underline{h}_t$ to be continuous at $z = d$ uniquely determines the fields above this plane when the fields below are known.

Assuming $\underset{\sim}{Z}_s$ and the fields in the region $z > d$ to be known, equation (12) is employed in finding the power flow and energy relations in this region. Equation (12) is valid for the fields in the region $z > d$ since the assumptions used in deriving it are satisfied in the present case -- see the text after (12). The differential quantities $d\underline{k}_t$ and $d\omega$ in the variation δ are arbitrary and independent since k_x, k_y and ω are independent variables. Integrating (12) from $z = d$ to $z = \infty$ gives

$$-j\frac{1}{2}\underline{z}_o \cdot (\underline{e}_t^* \times \delta\underline{h}_t + \delta\underline{e}_t \times \underline{h}_t^*)_{z=d} = W_s\, d\omega - d\underline{k}_t \cdot \underline{S}_s \tag{46}$$

where

$$W_s = \int_d^\infty w\, dz \tag{47}$$

[†]In the above relation, $-\underline{z}_o$ is used instead of $\underline{z}_o$, as was used in (28) and (32) for the medium above a perfectly conducting plane, because $-\underline{z}_o$ is the outward unit normal for the configuration being considered. The convention of using the outward unit normal in defining the impedance is based on the desire to have the impedance matrix be positive-definite when loss is present in the structure.

and the purely transverse vector $\underline{S}_s$ is

$$\underset{\sim}{S}_s = \int_d^\infty \underline{s}\, dz \quad . \tag{48}$$

In a manner similar to that of the previous section, the term on the left-hand side of (46) can be written in terms of $\delta \underset{\sim}{Z}_s$ if the anti-Hermitian property of $\underset{\sim}{Z}_s$ is taken into account. That $\underset{\sim}{Z}_s$ is anti-Hermitian follows from the fact that $s_z = 0$ for evanescent fields in a lossless, plane-stratified medium. In terms of $\delta \underset{\sim}{Z}_s$, (46) becomes

$$-j\frac{1}{2}\left[(-\underline{z}_o \times \underline{h}_t^*) \cdot \delta \underset{\sim}{Z}_s \cdot (\underline{z}_o \times \underline{h}_t)\right]_{z=d} = W_s d\omega - d\underline{k}_t \cdot \underset{\sim}{S}_s \quad . \tag{49}$$

Since dk_x, dk_y and $d\omega$ are all independent, the above relation can hold only if

$$\begin{aligned} &\Big[\underline{x}_o \frac{j}{2}(-\underline{z}_o \times \underline{h}_t^*) \cdot \frac{\partial \underset{\sim}{Z}_s}{\partial k_x} \cdot (-\underline{z}_o \times \underline{h}_t) \\ &\quad + \underline{y}_o \frac{j}{2}(-\underline{z}_o \times \underline{h}_t^*) \cdot \frac{\partial \underset{\sim}{Z}_s}{\partial k_y} \cdot (-\underline{z}_o \times \underline{h}_t)\Big]_{z=d} = \underline{S}_s \end{aligned} \tag{50}$$

and

$$-j\frac{1}{2}\left[(-\underline{z}_o \times \underline{h}_t^*) \cdot \frac{\partial \underset{\sim}{Z}_s}{\partial \omega} \cdot (-\underline{z}_o \times \underline{h}_t)\right]_{z=d} = W_s \quad . \tag{51}$$

The foregoing relations should be compared to (36) and (37). Note that if $\underline{z}_o$ instead of $-\underline{z}_o$ had been used in (45), the above relations would contain an additional minus sign.

The concept of a surface impedance to describe the effect of the medium above the plane $z = d$ on the fields below this plane can be employed to derive the dispersion relation for surface waves. The physical configuration to be considered here consists of a plane-stratified, lossless, anisotropic medium above the plane $z = d > 0$ and a second plane-stratified, lossless, anisotropic medium between a perfectly conducting plane at $z = 0$ and the plane $z = d$. It will be assumed that the values of $\underline{k}_t$ and ω of interest are such that the medium above the plane $z = d$ is representable in terms of an anti-Hermitian surface impedance $\underset{\sim}{Z}_s$ that satisfies (45). This restriction on $\underline{k}_t$ and ω is equivalent to the requirement that the fields in the region $z > d$ be of the surface wave

type for all $(\underline{h}_t)_{z=d}$. (In special cases, surface waves may exist when the fields in the region $z > d$ are of the surface wave type for only one polarization of $(\underline{h}_t)_{z=d}$. Such cases are not included in the present discussion.) The structure below the plane $z = d$ is assumed to be represented by the surface impedance $\underset{\sim}{Z}$ that satisfies (32).

Since $\underline{e}_t$ and $\underline{h}_t$ for the surface waves are continuous functions of z, these quantities must be the same in both (32) and (45). Thus subtracting these two equations gives

$$(\underset{\sim}{Z} + \underset{\sim}{Z}_s) \cdot (\underline{z}_o \times \underline{h}_t)_{z=d} = 0, \tag{52}$$

which is a homogeneous set of two linear equations in the two unknown elements of $(\underline{z}_o \times \underline{h}_t)_{z=d}$. For non-trivial solutions of (52) to exist one requires that $\det(\underset{\sim}{Z} + \underset{\sim}{Z}_s) = 0$, which gives the surface wave dispersion relation $D_s(\underline{k}_t, \omega) = 0$. At those values of $\underline{k}_t$ and ω which satisfy the surface wave dispersion relation, $(\underline{z}_o \times \underline{h}_t)_{z=d}$ can be found. If the partial derivatives of $\underset{\sim}{Z}$ and $\underset{\sim}{Z}_s$ with respect to k_x, k_y and ω are calculated, the power flow and stored energy can now be found in each region by using (36), (37), (50) and (51).

Thus it is seen that the knowledge of $\underset{\sim}{Z}$ and $\underset{\sim}{Z}_s$ for the lossless plane-stratified structures previously described is sufficient to find the surface wave dispersion relation and the division of power flow and stored energy between the two regions. Also, this procedure can be applied when the structure below the plane $z = d$ is a semi-infinite medium whose regions in $\underline{k}_t - \omega$ space, where the reactive surface impedance may be defined, intersect the corresponding regions for the medium above the plane $z = d$.

APPENDIX

In order to define that $\underset{\sim}{Z}$ which represents a plane-stratified, lossless, dispersive, anisotropic medium above a perfectly conducting plane, the auxiliary problem, with boundary conditions (26) and (27), was first considered. For most values of the parameters $\underline{k}_t$ and ω, the auxiliary problem will have unique non-trivial solutions of the form given in (1) for all $\underline{h}_d \neq 0$. At the remaining values of $\underline{k}_t$ and ω, which lie on surfaces in $\underline{k}_t - \omega$ space, non-unique cavity-type solutions will exist for $\underline{h}_d = 0$ and no solutions satisfying the boundary condition (27) will exist for all $\underline{h}_d \neq 0$. The non-unique solutions exist when the plane $z = d$ corresponds to a magnetic field null and would give infinite values for some or all of the components of the impedance dyadic $\underset{\sim}{Z}$. For this reason such values of $\underline{k}_t$ and ω are excluded from the consideration of $\underset{\sim}{Z}$. Although the surface impedance formalism breaks down at these values of $\underline{k}_t$ and ω, a surface admittance formalism will in general remain valid.

The surface admittance $\underset{\sim}{Y}$ is the inverse of $\underset{\sim}{Z}$, when both $\underset{\sim}{Z}$ and its inverse exist, and is regular at those values of $\underline{k}_t$ and ω for which $\underset{\sim}{Z}$ cannot be defined. In studying the properties of $\underset{\sim}{Y}$, one would consider the fields in the region $0 < z < d$ with $\underline{E}_t$, rather than $\underline{H}_t$, specified at $z = d$. Thus to find the admittance, one requires that $\underset{\sim}{Y}$ be such as to satisfy the relation

$$(\underline{z}_o \times \underline{h}_t)_{z=d} = \underset{\sim}{Y} \cdot (\underline{e}_t)_{z=d} \tag{53}$$

for two field solutions in the region $0 < z < d$. The two field solutions to be used are those satisfying the boundary condition

$$(\underline{E}_t)_{z=d} = \underline{e}_d \, e^{j(\omega t - \underline{k}_t \cdot \underline{\rho})} \tag{54}$$

at $z = d$ with $\underline{e}_d$ taking on two linearly independent forms. If (53) is satisfied for these two field solutions, because of linearity, it will be satisfied by the solutions for all possible $\underline{e}_d$. Those values of $\underline{k}_t$ and ω for which non-trivial solutions exist when $\underline{e}_d = 0$ are excluded from consideration. The impedance formalism may, however, be used, in general, at such values.

The energy and power relations containing $\underset{\sim}{Y}$ can be derived from reasoning similar to that used for $\underset{\sim}{Z}$. They are

$$\left.\begin{aligned} j\frac{1}{2}\left[\underline{e}_t^* \cdot \frac{\partial \underset{\sim}{Y}}{\partial k_x} \cdot \underline{e}_t\right]_{z=d} &= S_{dx} \\ j\frac{1}{2}\left[\underline{e}_t^* \cdot \frac{\partial \underset{\sim}{Y}}{\partial k_y} \cdot \underline{e}_t\right]_{z=d} &= S_{dy} \end{aligned}\right\} \tag{55}$$

and

$$-j\frac{1}{2}\left[\underline{e}_t^* \cdot \frac{\partial \underset{\sim}{Y}}{\partial \omega} \cdot \underline{e}_t\right]_{z=d} = W_d \quad . \tag{56}$$

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